Linear Algebra & Geometry

LECTURE 14

- Linear operators
- Change of basis matrix
- Eigenvalues, eigenvectors
- Diagonal matrices

Example. (From the previous lecture)
Consider $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) = (x + y, x - y)$. Find $M_S(\varphi)$ and $M_R(\varphi)$, where $S = \{(1,0), (0,1)\}$ and $R = \{(1,1), (2,1)\}$.

To find $M_R(\varphi)$:
$\varphi(1,1) = (2,0) = a(1,1) + b(2,1)$. Solving the system of equations $\begin{cases} a + 2b = 2 \\ a + b = 0 \end{cases}$ we get $a = -2, b = 2$.
$\varphi(2,1) = (3,1) = c(1,1) + d(2,1)$. Solving the system of equations $\begin{cases} c + 2d = 3 \\ c + d = 1 \end{cases}$ we get $c = -1, d = 2$. Hence, $M_R(\varphi) = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$.
Obviously, $M_S(\varphi) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (because $\varphi(1,0) = (1,1) = 1(1,0) + 1(0,1)$ and $\varphi(0,1) = (1, -1) = 1(1,0) + (-1)(0,1)$).

What is the relation between matrices of an operator in different bases? How can we tell that two matrices are matrices of the same operator but in different bases?

Theorem.

For every two bases R and S of $\mathbb{F}^n$ and for every linear operator φ on $\mathbb{F}^n$ there exists a matrix P such that

$$M_R^R(\varphi) = P^{-1}M_S^S(\varphi)P$$

Proof.

Let $P = M_S^R(id)$. Then $P^{-1} = M_R^S(id)$ (because $M_R^S(id)M_S^R(id) = M_R^R(id \circ id) = M_R^R(id) = I$). By the last theorem, $M_R^S(id)M_S^S(\varphi)M_S^R(id) = M_R^S(id)M_S^R(\varphi \circ id) = M_R^R(id \circ \varphi \circ id) = M_R^R(\varphi)$. QED

Note. The matrix P is called the *change-of-basis matrix*.

Definition.
Two $n \times n$ matrices A and B are said to be *similar* iff there exists a matrix P such that $A = P^{-1}BP$. We denote similarity of A and B by $A \approx B$.

Note. S*imilarity* should not be confused with *row equivalence*.

Fact.
Similarity of matrices is an equivalence relation on the set $\mathbb{K}^{n\times n}$.

Fact.
Two matrices are similar iff they are matrices of the same linear operator in two bases.

Example. (cont-d)

Consider the last example, $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) = (x + y, x - y)$, $S = \{(1,0), (0,1)\}$ and $R = \{(1,1), (2,1)\}$. It turned out that $A = M_R(\varphi) = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$ and $B = M_S(\varphi) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Verify that A and B are similar.

From the last theorem, the change-of-basis matrix is $M_S^R(id)$ (or $M_R^S(id)$, it works both ways, but $M_S^R(id)$ is easier to construct). This means we must represent vectors from R (transformed by id) as linear combinations of vectors from S. Nothing can be easier because S is the *unit vectors* basis. Hence, $[id(1,1)]_S = [(1,1)]_S = (1,1)$ and $[id(2,1)]_S = [(2,1)]_S = (2,1)$. Finally, $P = M_S^R(id) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. In order to verify this solution, we can check if $A = P^{-1}BP$. This requires finding P^{-1}.

Finally, $P = M_S^R(id) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. In order to verify this solution, we can check if $A = P^{-1}BP$. This requires finding P^{-1}:

$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim (r_1 - r_2) \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim (r_2 - r_1)$

$\begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix} \sim (r_2 \leftrightarrow r_1) \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$.

$BP = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$,

$P^{-1}BP = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$. We look-up our matrix A and … $A = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$, bingo!

FAQ.

Is it enough, instead of checking if $A = P^{-1}BP$ to check if $PA = BP$? It would save us the hassle of finding P^{-1}!

It depends. If you know for a fact that P is invertible then it is enough. Otherwise – not. For example, if P is the zero matrix then, obviously, $PA = BP$ while, equally obviously, P is not a change-of-basis matrix.

EIGENVALUES AND EIGENVECTORS

Definition.
Let A be an $n \times n$ matrix over $\mathbb{F}$. Every scalar λ such that for a nonzero vector X_λ from $\mathbb{F}^n$, $AX_\lambda = \lambda X_\lambda$ is called an *eigenvalue* of A and each X_λ is called an *eigenvector* belonging to λ.

These things have a wide range of applications from differential equations, through big data systems, through graph theory.

Notice.
Let φ be a linear operator. If for some scalar λ and for some vector X_λ, $\varphi(X_\lambda) = \lambda X_\lambda$ then λ and X_λ are also called an eigenvalue and an eigenvector of φ.

Example.

Find eigenvalues of $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix}$. (the corresponding $\varphi_A(x, y, z) = (-y + 2z, -2x - y + 4z, -2x - 2y + z))$.

This is equivalent to finding λ-s for whom there exists nonzero vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Moving the right-hand side to the left, we reduce the condition to the homogeneous system

$$\begin{cases} -\lambda a - b + 2c = 0 \\ -2a + (-1 - \lambda)b + 4c = 0 \\ -2a - 2b + (5 - \lambda)c = 0 \end{cases} \equiv (A - \lambda I) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nonzero solutions exist iff the rank of the coefficient matrix $A - \lambda I$ is smaller than its size, i.e., if $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ -2 & -2 & 5-\lambda \end{vmatrix} \underset{(r_3-r_2)}{=}$$

$$\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ 0 & -1+\lambda & 1-\lambda \end{vmatrix} \underset{taking\ out\ common\ factor\ of\ (1-\lambda)\ from\ r_3}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ 0 & -1 & 1 \end{vmatrix} \underset{(c_2+c_3)}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 & 2 \\ -2 & 3-\lambda & 4 \\ 0 & 0 & 1 \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} \underset{(r_2-r_1)}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 \\ \lambda-2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & 1 \end{vmatrix} =$$

$(1-\lambda)^2(2-\lambda) = 0$. Hence, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

Theorem.
A scalar λ is an eigenvalue for A iff $\det(A - \lambda \mathrm{I}) = 0$.

Proof. As in the example, instead of $AX = \lambda X$ we write $AX = (\lambda I)X$ which leads to $(A - \lambda I)X = \Theta$. Nonzero solutions to an $n \times n$ homogeneous system of equations exist iff the determinant of the coefficient matrix is zero.

Fact.
For every eigenvalue λ of A the set $W_\lambda = \{X \in \mathbb{K}^n | AX = \lambda X\}$ is a subspace in $\mathbb{K}^n$. The subspace is called an *eigenspace* for λ.

Proof. W_λ is the solution space for $(A - \lambda I)X = \Theta$.

We are on familiar grounds here; we know how to deal with homogeneous systems of equations. We have to do it separately for each eigenvalue, though.

Example – cont-d**.** Knowing that the eigenvalues are $\lambda_1, \lambda_2 = 1$ and $\lambda_3 = 2$, find eigenvectors of $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix}$ and the dimension of each eigenspace.

For λ =1 our system of equations reads $\begin{bmatrix} -1 & -1 & 2 \\ -2 & -2 & 4 \\ -2 & -2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $r_3 = r_2$ and $r_2 = 2r_1$ the rank of the matrix is 1 and the system is equivalent to $-a - b + 2c = 0$ which means $a = -b + 2c$ and b, c run free. So, all eigenvectors for λ =1 look like $(-b + 2c, b, c)$ = $b(-1,1,0) + c(2,0,1)$ and $\dim\left(W_{\lambda_1}\right) = 2.$

For λ =2 we get $\begin{bmatrix} -2 & -1 & 2 \\ -2 & -3 & 4 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Row reducing the matrix we obtain

$$\begin{bmatrix} -2 & -1 & 2 \\ -2 & -3 & 4 \\ -2 & -2 & 3 \end{bmatrix} \underset{r_2 - r_1, r_3 - r_1}{\sim} \begin{bmatrix} -2 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \underset{r_1 - r_3, r_2 - 2r_3}{\sim}$$

$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The rank is 2, dimension of the solution space is 1. We get $a - \frac{1}{2}c$=0 and $b - c = 0$, i.e., $c = 2a$ and $b = c = 2a$ and eigenvectors are $(a, 2a, 2a)$, for all $a \neq 0$ or $t(1,2,2)$ for all nonzero $t \in \mathbb{R}$. Hence, $\dim\left(W_{\lambda_2}\right) = 1$.

Fact.
Suppose $A \approx B$ with $A = P^{-1}BP$. Then

1. $\det(A) = \det(B)$ (*obvious*)

2. for every $n \in \mathbb{N}, A^n = P^{-1}B^nP$ (*obvious*)

3. $\det(A - \lambda I) = \det(B - \lambda I)$.

4. A and B have the same eigenvalues (*a consequence of 3*).

Proof (3). $A - \lambda I = P^{-1}BP - \lambda I = P^{-1}BP - \lambda P^{-1}IP = P^{-1}(B - \lambda I)$P hence $A - \lambda I \approx B - \lambda I$ so, by *1.*, their determinants are equal.

Corollary.
If λ is an eigenvalue of A then it is an eigenvalue of every matrix B similar to A.

Theorem.

Let φ be a linear operator and let $R = \{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{F}^n$. Then

$$M_R(\varphi) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$ ($M_R(\varphi)$ is a diagonal matrix) if and only if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of φ and $v_1, v_2, \dots, v_n$ are their eigenvectors.

Proof. ($\Leftarrow$)

For each $i = 1,2, \dots, n$, $\varphi(v_i) = \lambda_i v_i = 0v_1 + 0v_2 + \cdots + \lambda_i v_i + \cdots + 0v_n$. Hence, the i −th column of $M_R(\varphi)$ is $\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}$

(⇒)

Suppose $M_R(\varphi) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$. Then $[\varphi(v_i)]_R =$

$$M_R(\varphi)[v_i]_R = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = \lambda_i [v_i]_R$$

i.e., $\varphi(v_i) = \lambda_i v_i$. QED

Note. The existence of a basis consisting of eigenvectors is NOT guaranteed. For some matrices, similar diagonal matrix does not exist.

Theorem.
An $n \times n$ matrix A is similar to a diagonal matrix D iff there exists a basis of $\mathbb{F}^n$ consisting exclusively of eigenvectors of A. QED

Example - continued.

The theorem says that the matrix A from the last example,

$A=\begin{bmatrix}0 & -1 & 2\\ -2 & -1 & 4\\ -2 & -2 & 5\end{bmatrix}$ is similar to $D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$. We will find the change-of-basis matrix P. In slides 12 and 13 we found eigenvectors $(-1,1,0)$ and $(2,0,1)$ for eigenvalue 1 and for eigenvalue 2. Thus, the basis of eigenvectors, $R=\{(-1,1,0),$ $(2,0,1),(1,2,2)\}$. Since A is the matrix of φ in the standard basis, $D=M_R(\varphi)=M_R^S(id)AM_S^R(id)$ and we must only construct $\mathrm{P}=$ $M_S^R(id)$ which is easy: $id(-1,1,0)=(-1,1,0)=(-1)(1,0,0)+$ $1(0,1,0)+0(0,01)$ hence, the first column of P is $\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$. In the same way we get the second and the third column of P.

Example - continued.

$\det(P) = 0 + 1 + 0 - 0 + 2 - 4 = -1 \neq 0$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} -1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 4 \end{matrix} = PD$$

$$A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix} \begin{matrix} -1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 4 \end{matrix} = AP \qquad \text{Checks!}$$

(Here, instead of $D = P^{-1}AP$ we have verified $PD = AP$ because we know that P is invertible).